On iterated image size for point-symmetric relations

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Abstract

Let $\Gamma = (V, E)$ be a point-symmetric reflexive relation and let $v \in V$ such that $|\Gamma(v)|$ is finite (and hence $|\Gamma(x)|$ is finite for all x , by the transitive action of the group of automorphisms). Let $j \in \mathbb{N}$ be an integer such that $\Gamma^j(v) \cap \Gamma^-(v) = \{v\}$. Our main result states that

$$|\Gamma^j(v)| \geq |\Gamma^{j-1}(v)| + |\Gamma(v)| - 1.$$

As an application we have $|\Gamma^j(v)| \geq 1 + (|\Gamma(v)| - 1)j$. The last result confirms a recent conjecture of Seymour in the case of vertex-symmetric graphs. Also it gives a short proof for the validity of the Caccetta-Häggkvist conjecture for vertex-symmetric graphs and generalizes an additive result of Shepherdson.

1 Introduction

Let G be an abelian group and let A, S be finite subsets of G with $0 \notin S$. Shepherdson's generalization of the Cauchy-Davenport Theorem states that $|A \cup (A + S)| \geq |A| + |S|$ if $A \cup (A + S)$ contains no subgroup generated by some element of S.

As an application Shepherdson [14] proved that there are $s_1, \cdots, s_k \in S$ such that $k \leq \lceil \frac{|G|}{|S|} \rceil$ and $\sum_{1 \leq i \leq k} s_i = 0$, if G is finite. The paper of Shepherdson includes thanks to Heilbronn for suggesting this application together with a mention that Chowla obtained some related zero-sum results.

Let $D = (V, E)$ be a loopless finite digraph with minimal outdgree at least 1. It is well known that D contains a directed cycle. The smallest cardinality of such a cycle is called the girth of D and will be denoted by $g(D)$. In 1970 Behzad, Chartrand and Wall [1] conjectured that $|V| \geq r(g(D) - 1) + 1$, if $d^+(x) = d^-(x) = r$ for all $x \in V$. In 1978, Caccetta and Häggkvist [3] made the stronger conjecture :

$$|V| \geq \min(d^+x : x \in V)(g(D) - 1) + 1.$$

These conjectures are still largely open, even for the special case $g(D) = 4$. The reader may find references and results about this question in [2].

These conjectures were proved by the author for vertex-symmetric digraphs [6]. This result applied to Cayley graphs shows the validity of Shepherdson's zero-sum result for all finite groups. Unfortunately we were not aware at that moment of Shepherdson's result. Our

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proof [6] is based on the properties of atoms of a finite digraph and Menger's Theorem. A description of Cayley graphs on finite Abelian groups such that $|V| = r(g(D) - 1) + 1$, where r is the outdegree was obtained by the authors of [9] using Kemperman critical pair Theory [12]. A new proof of the Caccetta and Häggkvist conjecture for vertex-symmetric digraphs based on an additive result of Kemperman [11] and the representation of vertex symmetric digraphs as coset graphs is given in [10].

More recently Seymour proposed the following conjecture [13]:

Let D be a loopless digraph and let $r \geq 1$ be an integer. Then there is a vertex a such that

$$|\Gamma(a) \cup \Gamma^2(a) \cup \cdots \cup \Gamma^{g-2}(a)| \geq r(g-2),$$

where $g = g(D)$.

The case $g = 4$ of this conjecture is mentioned in [2]. Seymour's Conjecture implies the conjecture of Behzad, Chartrand and Wall. Seymour's Conjecture also implies that D contains a directed cycle C with $|C| \leq \lceil \frac{|V|-1}{r} \rceil + 1$. Notice that the Caccetta-Häggkvist Conjecture states that D contains a directed cycle C with $|C| \leq \lceil \frac{|V|}{r} \rceil$.

We shall allow infinite relations. The classical strong connectivity of digraphs needs to be modified in this case in order to have a good lower bound of the size of the image of a set. Also the presence of loops will simplify the presentation of the connectivity method. Since this convention is unusual in this part of Graph Theory, we shall work with relations. Our terminology will be developed in the next section.

Seymour's conjecture may be formulated as follows :

Conjecture 1 *[13] Let $\Gamma = (V, E)$ be a finite reflexive relation and let j be an integer. Then there is an $x \in V$ such that one of the following conditions holds.*

- $|\Gamma^j(x)| \geq 1 + j(|\Gamma(x)| - 1)$.
- $\Gamma^{-1}(x) \cap \Gamma^j(x) \neq \{x\}$.

Our main result is the following one:

Let $\Gamma = (V, E)$ be a point-symmetric reflexive relation and let $v \in V$ such that $|\Gamma(v)|$ is finite. Let $j \geq 1$ be such that $\Gamma^j(v) \cap \Gamma^-(v) = \{v\}$. Then

$$|\Gamma^j(v)| \geq |\Gamma^{j-1}(v)| + |\Gamma(v)| - 1.$$

This result implies the validity of the above conjectures for vertex-symmetric graphs.

2 Terminology

Let V be a set. The diagonal of V is by definition $\Delta_V = \{(x, x) : x \in V\}$. Let $E \subset V \times V$. The ordered pair $\Gamma = (V, E)$ will be called a *relation* . The relation Γ is said to be *reflexive* if $\Delta_V \subset E$.

Let $a \in V$ and let $A \subset V$. The image of a is by definition

$$\Gamma(a) = \{x : (a, x) \in E\}.$$

The image of A is by definition

$$\Gamma(A) = \bigcup_{x \in A} \Gamma(x).$$

The cardinality of the image of x will be called the *degree* of x and will be denoted by $d(x)$. The relation Γ will be called *regular* with degree r if the elements of V have the same degree r. We shall say that Γ is *locally finite* if $d(x)$ is finite for all x. The *reverse* relation of Γ is by definition $\Gamma^- = (V, E^-)$, where $E^- = \{(x, y) \Big| \ (y, x) \in E\}$. The *restriction* of $\Gamma = (V, E)$ to a subset $W \subset V$ is defined as the relation $\Gamma[W] = (W, E \cap (W \times W))$.

Let $\Phi = (W, F)$ be a relation. A function $f : V \longrightarrow W$ will be called a *homomorphism* if for all $x, y \in V$ such that $(x, y) \in E$, we have $(f(x), f(y)) \in F$.

The relation Γ will be called *point-symmetric* if for all $x, y \in V$, there is an automorphism f such that $y = f(x)$. Clearly a point-symmetric relation is regular.

We identify graphs and their relations. A loopless finite relation will be called a *digraph*. The reader may replace everywhere the term "relation" by "graph". In this case we mention some differences between our terminology (which follows closely the standard notations of Set Theory) and the notations used in some text books of Graph Theory. We point out that our graphs are usually called directed graphs without multiple arcs or digraphs. Notice that the notion $\Gamma(a)$ used here and in Set Theory is written $\Gamma^+(a)$ in some text books in Graph Theory. Also our notion of degree is called *outdegree*. We made the choice of Set Theory terminology since some parts of this paper could have some interest in Group Theory and Number Theory.

We shall use the composition $\Gamma_1 \circ \cdots \circ \Gamma_k$ of relations $\Gamma_1, \cdots, \Gamma_k$ on V. If all these relations are equal to Γ, we shall write

$$\Gamma_1 \circ \cdots \circ \Gamma_k = \Gamma^k.$$

We shall write Γ^0 for the identity relation $I_V = (V, \Delta_V)$. Also we shall write Γ^{-j} instead of $(\Gamma^-)^j$.

3 Connectivity

Let $\Gamma = (V, E)$ be a relation. For $X \subset V$, we shall write

$$\partial_\Gamma(X) = \Gamma(X) \setminus X.$$

When the context is clear the reference to Γ will be omitted.

Let $\Gamma = (V, E)$ be a locally finite reflexive relation. The *connectivity* of Γ is by definition $\kappa(\Gamma) = |V| - 1$, if $E = V \times V$. Otherwise

$$\kappa(\Gamma) = \min\{|\partial(X)| \ : \ 1 \leq |X| < \infty \text{ and } \Gamma(X) \neq V\}. \tag{1}$$

A subset X achieving the minimum in (1) is called a *fragment* of Γ. A fragment with minimum cardinality is called an *atom*. The cardinality of an atom of Γ will be denoted by $a(\Gamma)$. It is not true that distinct atoms are always disjoint. But the author proved in [4] that, if V is finite, then distinct atoms of Γ are disjoint, or distinct atoms of Γ^- are disjoint. In [7], it was observed that the same methods imply that distinct atoms of Γ are disjoint if V is infinite. One may find in [8] unified proofs and some applications to Group Theory and Additive Number Theory.

As a consequence of this result we could obtain :

Proposition 2 *[4, 5, 7, 8] Let $\Gamma = (V, E)$ be a locally-finite point-symmetric relation with $E \neq V \times V$. Suppose that V is infinite or that $a(\Gamma) \leq a(\Gamma^-)$. Let A be an atom of Γ. Then the subrelation $\Gamma[A]$ induced on A is a point-symmetric relation. Moreover $|A| \leq \kappa(\Gamma)$.*

4 Iterated image size

Lemma 3 *Let $\Gamma = (V, E)$ be a point-symmetric relation. Then for all i, Γ^i is point-symmetric.*

Proof. Clearly any automorphism of Γ is an automorphism of Γ^i. ∎

Theorem 4 *Let $\Gamma = (V, E)$ be a point-symmetric reflexive locally finite relation and let $v \in V$. Let $j \geq 1$ be an integer such that $\Gamma^j(v) \cap \Gamma^-(v) = \{v\}$. Then*

$$|\Gamma^j(v)| \geq |\Gamma^{j-1}(v)| + |\Gamma(v)| - 1.$$

Proof.

Set $V_0 = \bigcup_{0 \leq i} \Gamma^i(v)$. Clearly $\Gamma^j(v) \subset V_0$. So we may assume that $\Gamma = \Gamma[V_0]$ and $V = V_0$.

In the finite case this means that we restrict ourselves to the connected component containing v.

We shall assume $j > 1$, since the result is obvious for $j = 1$.

With this hypothesis, clearly we have $\kappa(\Gamma) \geq 1$.

Clearly

$$E \neq V \times V.$$

Set $\kappa = \kappa(\Gamma)$. Let A be an atom of Γ containing v. The proof is by induction on $|\Gamma(v)|$.

Put $r = |\Gamma(v)|$. Assume first

$$\kappa = r - 1.$$

Observe that $\Gamma^j(v) \neq V$. Then by the definition of κ, we have

$$|\Gamma^j(v) \setminus \Gamma^{j-1}(v)| = |\partial(\Gamma^{j-1}(v))| \geq \kappa = r - 1.$$

The result holds in this case. So we may assume

$$\kappa \leq r - 2, \tag{2}$$

and hence $r \geq 3$. Then $|A| \geq 2$, since otherwise $\kappa = |\partial(A)| = r - 1$.

Case 1. V is infinite or $a(\Gamma) \leq a(\Gamma^{-})$.

By Proposition 2, $\Gamma[A]$ is point-symmetric (and hence regular) and

$$|A| \leq \kappa. \tag{3}$$

Put $r_0 = |\Gamma(v) \cap A|$.

Put $X = \Gamma^{j-1}(v)$ and $Y = A \cup X$. By the induction hypothesis, we have

$$|\partial(X) \cap A| = |\Gamma^{j}(v) \cap A| - |\Gamma^{j-1}(v) \cap A| \geq r_0 - 1. \tag{4}$$

Let us prove that

$$\partial(Y) \subset (\partial(X) \setminus A) \cup (\partial(A) \setminus \Gamma(v)). \tag{5}$$

Since $j \geq 2$, we have $\Gamma(v) \subset X$ and hence $\Gamma(v) \cap \partial(X) = \emptyset$. Then (5) clearly holds.

It follows that

$$\partial(Y) \setminus \partial(X) \subset \partial(A) \setminus \Gamma(v),$$

and hence we have

$$\begin{aligned} |\partial(Y) \setminus \partial(X)| &\leq |\partial(A)| - |\partial(A) \cap \Gamma(v)| \\ &= \kappa - |\Gamma(v)| + |A \cap \Gamma(v)| \\ &= \kappa + r_0 - r. \end{aligned}$$

Hence

$$|\partial(Y) \setminus \partial(X)| \leq \kappa + r_0 - r. \tag{6}$$

Let us show that $\Gamma(Y) \neq V$. This holds obviously if V is infinite. So we may assume V finite. In this case we have $|\Gamma^{-}(v)| = |\Gamma(v)|$.

Clearly we have

$$\Gamma(Y) = \Gamma(X) \cup (A \setminus \Gamma(v)) \cup (\partial(Y) \setminus \partial(X)).$$

It follows using (3) and (6) that

$$\begin{aligned} |\Gamma(Y)| &\leq |\Gamma(X)| + |A \setminus \Gamma(v)| + |\partial(Y) \setminus \partial(X)| \\ &\leq |V \setminus (\Gamma^{-}(v) \setminus \{v\})| + |A| - r_0 + \kappa + r_0 - r \\ &= |V| + |A| + \kappa - 2r + 1 \\ &\leq |V| + 2\kappa - 2r + 1 \leq |V| - 3. \end{aligned}$$

By the definition of κ, we have $|\partial(Y)| \geq \kappa$.

By (4) and (6),

$$\begin{array}{rcl} |\partial(X)| & = & |\partial(X) \cap A| + |\partial(Y) \cap \partial(X)| \\ & \geq & r_0 - 1 + |\partial(Y)| - |\partial(Y) \setminus \partial(X)| \\ & \geq & r_0 - 1 + \kappa - (\kappa + r_0 - r) = r - 1, \end{array}$$

and the result is proved since

$$\partial(X) = \Gamma^j(v) \setminus \Gamma^{j-1}(v).$$

Case 2. V is finite and $a(\Gamma) > a(\Gamma^-)$.

The argument used in Case 1, shows that $|\Gamma^{-j}(v) \setminus \Gamma^{-(j-1)}(v)| \geq r - 1$.

By Lemma 3, Γ^j is point-symmetric. Since V is finite, Γ^j and its reverse Γ^{-j} have the same degree. Therefore observing that these relations are reflexive

$$\begin{array}{rcl} r - 1 & \leq & |\Gamma^{-j}(v)| - |\Gamma^{-(j-1)}(v)| \\ & = & |\Gamma^j(v)| - |\Gamma^{(j-1)}(v)|. \end{array}$$

∎

The next result shows the validity of the conjecture of Seymour mentioned in the introduction in the case of relations with a symmetric group of automorphisms.

Corollary 5 *Let $\Gamma = (V, E)$ be a point-symmetric reflexive relation with degree r and let $v \in V$. Let $j \geq 1$ be an integer such that $\Gamma^j(v) \cap \Gamma^-(v) = \{v\}$ then*

$$|\Gamma^j(v)| \geq 1 + (r-1)j.$$

Proof. The proof follows by induction using Theorem 4

∎

Corollary 6 *[6] Let $\Gamma = (V, E)$ be a point-symmetric digraph with degree $r \geq 1$ and put $g = g(\Gamma)$. Then $|V| \geq 1 + r(g-1)$.*

Proof. Set $\Phi = (V, E \cup \Delta_V)$. Let $v \in V$. Clearly we have $\Phi^{g-2}(v) \cap \Phi^-(v) = \{v\}$. By Corollary 5, $|V| - r = |V \setminus (\Phi^-(a) \setminus \{a\})| \geq |\Phi^{g-2}(v)| \geq 1 + (g-2)r$. ∎

This result, proved in [6], shows the validity of the Caccetta-Häggkvist Conjecture for point-symmetric graphs. But the proof obtained here is much easier.

Corollary 7 *[6] Let G be a group of order n and let $S \subset G \setminus \{1\}$ with cardinality $= s$. There are elements $s_1, s_2, \cdots, s_k \in S$ such that $k \leq \lceil \frac{n}{s} \rceil$ and $\prod_{1 \leq i \leq k} s_i = 1$.*

The proof follows by applying Corollary 6 to the Cayley graph defined by S on G. In particular the theorem of Shepherdson mentioned in the introduction holds for all finite groups.

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